A bijection between the irreducible k-shapes and the surjective pistols of height k-1

Ange Bigeni

A BIJECTION BETWEEN IRREDUCIBLE k-SHAPES THE SURJECTIVE PISTOLS OF HEIGHT $k-1$

ANGE BIGENI

Abstract. This paper constructs a bijection between irreducible k-shapes and surjective pistols of height $k-1$, which carries the "free k-sites" to the fixed points of surjective pistols. The bijection confirms a conjecture of Hivert and Mallet (FPSAC 2011) that the number of irreducible k-shape is counted by the Genocchi number G_{2k}.

1. Introduction

The study of k-shapes arises naturally in the combinatorics of k-Schur functions (see [LLMS13]). In a 2011 FPSAC paper, Hivert and Mallet showed that the generating function of all k-shapes was a rational function whose numerator $P_k(t)$ was defined in terms of what they called irreducible k-shapes. The sequence of numbers of irreducible k-shapes $(P_k(1))_{k\geq 1}$ seemed to be the sequence of Genocchi numbers $(G_{2k})_{k\geq 1} = (1, 1, 3, 17, 155, 2073, \dots)$ [OEI], which may be defined by $G_{2k} = Q_{2k-2}(1)$ for all $k \geq 2$ (see [Car71, RS73]) where $Q_{2n}(x)$ is the Gandhi polynomial defined by the recursion $Q_2(x) = x^2$ and

$$Q_{2k+2}(x) = x^2(Q_{2k}(x+1) - Q_{2k}(x)). \tag{1}$$

Hivert and Mallet defined a statistic $fr(\lambda)$ counting the so-called free k-sites on the partitions λ in the set of irreducible k-shapes IS_k, and conjectured that

$$Q_{2k-2}(x) = \sum_{\lambda \in IS_k} x^{fr(\lambda)+2}. \tag{2}$$

The goal of this paper is to construct a bijection between irreducible k-shapes and surjective pistols of height $k-1$, such that every free k-site of an irreducible k-shape is carried to a fixed point of the corresponding surjective pistol. Since the surjective pistols are known to generate the Gandhi polynomials with respect to the fixed points (see Theorem 2.1), this bijection will imply Formula (2).

The rest of this paper is organized as follows. In Section 2, we give some background about surjective pistols (in Subsection 2.1), partitions, skew partitions and k-shapes (in Subsection 2.2), then we focus on irreducible k-shapes (in Subsection 2.3) and enounce Conjecture 2.1 raised by Mallet (which implies Formula 2), and the main result of this paper, Theorem 2.2, whose latter conjecture is a straight corollary. In Section 3, we give preliminaries of the proof of Theorem 2.2 by introducing the notion of partial k-shapes. In Section 4, we demonstrate Theorem 2.2 by defining two inverse maps φ (in Subsection 4.1) and ϕ (in Subsection 4.2) which connect irreducible k-shapes and surjective pistols and keep track of the two statistics. Finally, in Section 5, we explore the corresponding interpretations of some generalizations of the Gandhi polynomials, generated by the surjective pistols with respect to refined statistics, on the irreducible k-shapes.

2. Definitions and main result

2.1. **Surjective pistols.** For all positive integer n, we denote by $[n]$ the set $\{1, 2, \dots, n\}$. A *surjective pistol of height* k is a surjective map $f : [2k] \to \{2, 4, \dots, 2k\}$ such that $f(j) \geq j$ for

Key words and phrases. Genocchi numbers; Gandhi polynomials; (irreducible) k-shapes; surjective pistols.

all $j \in [2k]$. We denote by SP_k the set of surjective pistols of height k. By abuse of notation, we assimilate a surjective pistol $f \in SP_k$ into the sequence $(f(1), f(2), \dots, f(2k))$. A *fixed point* of $f \in SP_k$ is an integer $j \in [2k]$ such that $f(j) = j$. We denote by $fix(f)$ the number of fixed points different from $2k$ (which is always a fixed point). A surjective pistol of height k can also be seen as a tableau made of k right-justified rows of length $2, 4, 6, \dots, 2k$ (from top to bottom), such that each row contains at least one dot, and each column contains exactly one dot. The map f corresponding to such a tableau would be defined as $f(j) = 2(\lceil j/2 \rceil + z_j)$ where the j-th column of the tableau contains a dot in its $(1 + z_j)$-th cell (from top to bottom) for all $j \in [2k]$. For example, if $f = (2, 4, 4, 8, 8, 6, 8, 8) \in SP_4$, the tableau corresponding to f is depicted in Figure 1.

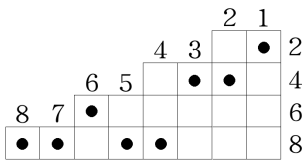

FIGURE 1. Tableau corresponding to $f = (2, 4, 4, 8, 8, 6, 8, 8) \in SP_4$.

In particular, an integer $j = 2i$ is a fixed point of f if and only if the dot of the $2i$-th column of the corresponding tableau is at the top of the column. For example, the surjective pistol f of Figure 1 has 2 fixed points 6 and 8, but $fix(f) = 1$ (because the fixed point $2k = 8$ is not counted by the statistic). The following result is due to Dumont.

Theorem 2.1 ([Dum74])**.** *For all $k \geq 2$, the Gandhi polynomial $Q_{2k}(x)$ has the following combinatorial interpretation:*

$$Q_{2k}(x) = \sum_{f \in SP_k} x^{fix(f)+2}.$$

2.2. **Partitions, skew partitions, k-shapes.** A partition is a a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. By abuse of definition, we consider that a partition may be empty (corresponding to $m = 0$). A convenient way to visualize a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is to consider its Ferrers diagram (denoted by $[\lambda]$), which is composed of cells organized in left-justified rows such that the i-th row (from bottom to top) contains λ_i cells. The *hook length* of a cell c is defined as the number of cells located to its right in the same row (including c itself) or above it in the same column. If the hook length of a cell c equals h, we say that c is hook lengthed by the integer h. For example, the Ferrers diagram of the partition $\lambda = (4, 2, 2, 1)$ is represented in Figure 2, in which every cell is labeled by its own hook length.

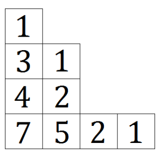

FIGURE 2. Ferrers diagram of the partition $\lambda = (4, 2, 2, 1)$.

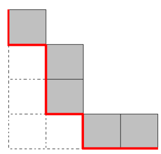

FIGURE 3. Skew partition $\lambda \backslash \mu$.

We will sometimes assimilate partitions with their Ferrers diagrams. If two partitions $\lambda = (\lambda_1, \dots, \lambda_p)$ and $\mu = (\mu_1, \dots, \mu_q)$ (with $q \leq p$) are such that $\mu_i \leq \lambda_i$ for all $i \leq q$, then we write $\mu \subseteq \lambda$ and we define the *skew partition* $s = \lambda/\mu$ as the diagram $[\lambda] \backslash [\mu]$, the Ferrers diagram $[\mu]$ appearing naturally in $[\lambda]$. For example, if $\lambda = (4, 2, 2, 1)$ and $\mu = (2, 1, 1)$, then $\mu \subseteq \lambda$ and the skew partition $\lambda \backslash \mu$ is the diagram depicted in Figure 3. For all skew partition s, we

name *row shape* (respectively *column shape*) of s, and we denote by $rs(s)$ (resp. $cs(s)$), the sequence of the lengths of the rows from bottom to top (resp. the sequence of the heights of the columns from left to right) of s. Those sequences are not necessarily partitions. For example, if s is the skew partition depicted in Figure 3, then $rs(s) = (2, 1, 1, 1)$ and $cs(s) = (1, 2, 1, 1)$ (in particular $cs(s)$ is not a partition). If the lower border of s is continuous, *i.e.*, if it is not fragmented into several pieces,, we also define a canonical partition $< s >$ obtained by inserting cells in the empty space beneath every column and on the left of every row of s. For example, if s is the skew partition depicted in Figure 3, the lower border of s is drawed as a thin red line which is continuous, and $< s >$ is simply the original partition $\lambda = (4, 2, 2, 1)$.
Now, consider a positive integer k. For all partition λ, it is easy to see that the diagram composed of the cells of $[\lambda]$ whose hook length does not exceed k, is a skew partition, that we name *k-boundary* of λ and denote by $\partial^k(\lambda)$. Incidentally, we name *k-rim* of λ the lower border of $\partial^k(\lambda)$ (which may be fragmented), and we denote by $rs^k(\lambda)$ (respectively $cs^k(\lambda)$) the sequence $rs(\partial^k(\lambda))$ (resp. the sequence $cs(\partial^k(\lambda))$. For example, the 2-boundary of the partition $\lambda = (4, 2, 2, 1)$ depicted in Figure 2, is in fact the skew partition of Figure 3. Note that if the k-rim of λ is continuous, then the partition $< \partial^k(\lambda) >$ is simply λ.

Definition 2.1 ([LLMS13])**.** A k-shape is a partition λ such that the sequences $rs^k(\lambda)$ and $cs^k(\lambda)$ are also partitions.

For example, the partition $\lambda = (4, 2, 2, 1)$ depicted in Figure 2 is not a 2-shape since $cs^2(\lambda) = (1, 2, 1, 1)$ is not a partition, but it is a k-shape for any $k \geq 4$ (for instance $cs^5(\lambda) = (3, 3, 1, 1)$ and $rs^5(\lambda) = (3, 2, 2, 1)$ are partitions, so λ is a 5-shape, see Figure 4). Note that the k-rim of a k-shape λ is necessarily continuous, thence $\lambda =< \partial^k(\lambda) >$. Consequently, we will sometimes assimilate a k-shape into its k-boundary.

2.3. **Irreducible k-shapes.** Let λ be a k-shape and (u, v) a pair of positive integers. Following [HM11], we denote by $H_u(\lambda)$ (respectively $V_v(\lambda)$) the set of all cells of the skew partition $\partial^k(\lambda)$ that are contained in a row of length u (resp. the set of all cells of $\partial^k(\lambda)$ that are contained in a column of height v). For example, consider the 5-shape $\lambda = (4, 2, 2, 1)$. The sets $(H_u(\lambda))_{u \geq 1}$

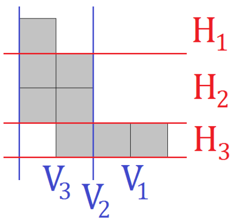

FIGURE 4. Skew partition $\partial^5(\lambda)$ with $\lambda = (4, 2, 2, 1)$.

and $(V_v(\lambda))_{v \geq 1}$ are outlined in Figure 4 (in this example the set $V_2(\lambda)$ is empty). Note that for all k-shape λ and for all pair of positive integers (u, v), if the set $H_u(\lambda) \cap V_v(\lambda)$ is not empty, then there exists a cell in $V_v(\lambda)$ hook lengthed by at least $u + v - 1$. Consequently, if $u + v > k + 1$, then by definition of $\partial^k(\lambda)$ the set $H_u(\lambda) \cap V_v(\lambda)$ must be empty.
Hivert and Mallet [HM11] defined an operation which consists in inserting, in a k-shape, a l-rectangle (namely, a partition whose Ferrers diagram is a rectangle and whose largest hook length is l) with $l \in \{k - 1, k\}$, the result of the operation being a new k-shape. They defined *irreducible k-shapes* as k-shapes that cannot be obtained in such a way. In this paper, we use an equivalent definition in view of the Proposition 3.8 of [HM11].

Definition 2.2 ([HM11])**.** An *irreducible k-shape* is a k-shape λ such that the sets $H_i(\lambda) \cap V_{k-i}(\lambda)$ and $H_i(\lambda) \cap V_{k+1-i}(\lambda)$ contain at most $i - 1$ horizontal steps of the k-rim of λ for all $i \in [k]$. We denote by IS_k the set of irreducible k-shapes.

For example, the 5-shape $\lambda = (4,2,2,1)$ (see Figure 4) is irreducible: the sets $H_i(\lambda) \cap V_{5-i}(\lambda)$ and $H_j(\lambda) \cap V_{6-j}(\lambda)$ are empty if $i \neq 2$ and $j \neq 3$, and the two sets $H_2(\lambda) \cap V_3(\lambda)$ and $H_3(\lambda) \cap V_3(\lambda)$ contain respectively $1 < 2$ and $1 < 3$ horizontal steps of the k-rim of λ.
In general, it is easy to see that for any k-shape λ to be irreducible, the sets $H_1(\lambda) \cap V_k(\lambda)$ and $H_k(\lambda) \cap V_1(\lambda)$ must be empty, and by definition the set $H_1(\lambda) \cap V_{k-1}(\lambda)$ must contain no horizontal step of the k-rim of λ. In particular, for $k = 1$ or 2 there is only one irreducible k-shape: the empty partition.

Definition 2.3 ([HM11, Mal11])**.** Let λ be an irreducible k-shape with $k \geq 3$. For all $i \in [k-2]$, we say that the integer i is a *free k-site* of λ if the set $H_{k-i}(\lambda) \cap V_{i+1}(\lambda)$ is empty. We define $\overrightarrow{fr}(\lambda)$ as the vector $(t_1, t_2, \ldots, t_{k-2}) \in \{0,1\}^{k-2}$ where $t_i = 1$ if and only if i is a free k-site of λ. We also define $fr(\lambda)$ as $\sum_{i=1}^{k-2} t_i$ (the quantity of free k-sites of λ).

For example, the irreducible 5-shape $\lambda = (4,2,2,1)$ depicted in Figure 4 is such that $\overrightarrow{fr}(\lambda) = (1,0,1)$. In order to prove the conjecture of Formula 2, and in view of Theorem 2.1, Hivert and Mallet proposed to construct a bijection $\phi : IS_k \to SP_{k-1}$ such that $fix(\phi(\lambda)) = fr(\lambda)$ for all λ. Mallet [Mal11] refined the conjecture by introducing a vectorial version of the statistic of fixed points: for all $f \in SP_{k-1}$, we define $\overrightarrow{fix}(f)$ as the vector $(t_1, \ldots, t_{k-2}) \in \{0,1\}^{k-2}$ where $t_i = 1$ if and only if $f(2i) = 2i$ (in particular $fix(f) = \sum_i t_i$).

Conjecture 2.1 ([Mal11])**.** *For all $k \geq 3$ and $\overrightarrow{v} = (v_1, v_2, \ldots, v_{k-2}) \in \{0,1\}^{k-2}$, the number of irreducible k-shapes λ such that $\overrightarrow{fr}(\lambda) = \overrightarrow{v}$ is the number of surjective pistols $f \in SP_{k-1}$ such that $\overrightarrow{fix}(f) = \overrightarrow{v}$.*

The main result of this paper is the following theorem, which implies immediately Conjecture 2.1.

Theorem 2.2. *There exists a bijection $\varphi : SP_{k-1} \to IS_k$ such that $\overrightarrow{fr}(\varphi(f)) = \overrightarrow{fix}(f)$ for all $f \in SP_{k-1}$.*

We intend to demonstrate Theorem 2.2 in the following two sections §3 and §4.

3. Partial k-shapes

Definition 3.1 (Labeled skew partitions, partial k-shapes and saturation property)**.** A *labeled skew partition* is a skew partition s whose columns are labeled by the integer 1 or 2. If $cs(s)$ is a partition and if the hook length of every cell of s doesn't exceed k (resp. $k-1$) when the cell is located in a column labeled by 1 (resp. by 2), we say that s is a *partial k-shape*. In that case, if C_0 is a column labeled by 1 which is rooted in a row R_0 (*i.e.*, whose bottom cell is located in R_0) whose top left cell is hook lengthed by k, we say that C_0 is saturated. For all $i \in [k-1]$, if every column of height $i+1$ and label 1 is saturated in s, we say that s is saturated in i. If s is saturated in i for all i, we say that s is saturated.

We represent labeled skew partitions by painting in dark blue columns labeled by 1, and in light blue columns labeled by 2. For example, the skew partition depicted in Figure 5 is a partial 6-shape, which is not saturated because its unique column C labeled by 1 is rooted in a row whose top left cell (which is in this exemple the own bottom cell of C) is hook lengthed by 5 instead of 6.

Definition 3.2 (Sum of partial k-shapes with rectangles)**.** Let s be a partial k-shape, and $j \geq 1$ such that the height of every column of s is at least $\lceil (j+2)/2 \rceil$ (if s is the empty skew partition we impose no condition on j). Let z be a nonnegative integer and $t(j)$ the integer defined as 1 if j is even and 2 if j is odd. We consider the labeled skew partition $\tilde{s}$ obtained by gluing right on the last column of s, the amount of z columns of height $\lceil (j+1)/2 \rceil$ (see

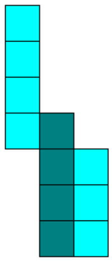

FIGURE 5. Partial 6-shape s.

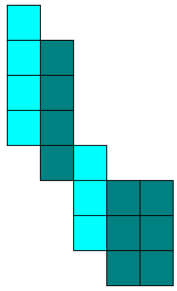

FIGURE 6. Partial 6-shape $s \oplus_1^6 3^2$.

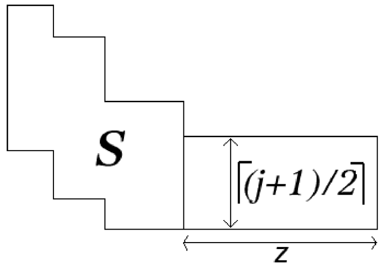

FIGURE 7. Gluing of the rectangle $\lceil (j+1)/2 \rceil^{z_j}$ to s.

Figure 7) labeled by the integer $t(j)$. We apply the following algorithm on $\tilde{s}$ as long as one of the three corresponding conditions is satisfied.

(1) If there exists a column C_0 labeled by 1 (respectively by 2) in $\tilde{s}$ such that the bottom cell c_0 of C_0 is a corner of $\tilde{s}$ (a cell of $\tilde{s}$ with no other cell beneath it or on the left of it) whose hook length h exceeds k (resp. $k-1$), then we *lift* the column C_0, *i.e.*, we erase c_0 and we draw a cell on the top of C_0 (see Figure 8).

(2) If there exists a column C_0 of height i_0+1 (with $i_0 \in [k-2]$) and labeled by 1 in $\tilde{s}$, such that the bottom cell c_0 of C_0 is on on the right of the bottom cell of a column whose height is not i_0+1 or whose label is not 1, then we lift every column on the left of C_0 whose bottom cell is located in the same row as c_0, *i.e.*, we erase every cell on the left of c_0 and we draw a cell on every corresponding column (see Figure 9).

(3) If there exists a column C_0 of height i_0+1 (with $i_0 \in [k-2]$) and labeled by 1 in $\tilde{s}$, such that the bottom cell c_0 of C_0 is a corner whose hook length h doesn't equal k (which means it is rooted in a row R_0 of $\tilde{s}$ whose length is $k - i_0 - l < k - i_0$ for some $l \geq 1$), whereas this hook length was exactly k in the partial k-shape s, then we lift every column rooted in the same row as the l last columns (from left to right) intersecting R_0, in such a way the hook length of c_0 becomes k again in $\tilde{s}$ (see Figure 10).

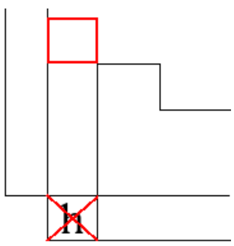

FIGURE 8

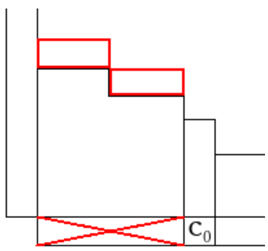

FIGURE 9

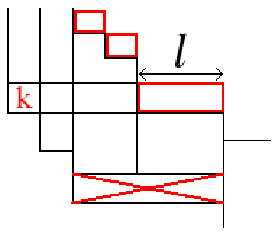

FIGURE 10

It is easy to see that this algorithm is finite and that the final version of $\tilde{s}$ is a partial k-shape, that we define as the $t(j)$-sum of the partial k-shape s with the rectangle $\lceil (j+1)/2 \rceil^z$ (the partition whose Ferrers diagram is a rectangle of length z and height $\lceil (j+1)/2 \rceil$), and which we denote by

$$s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z.$$

For example, the 1-sum $s \oplus_1^6 3^2$ of the partial 6-shape s represented in Figure 5, with the rectangle composed of 2 columns of height 3 and label 1, is the partial 6-shape depicted in Figure 6.

Remark 3.1. In the context of Definition 3.2, the rule (3) of the latter definition guarantees that any saturated column of s is still saturated in $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$. In particular, if s is saturated in $i \in [k-2]$, then $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$ is also saturated in i.

Lemma 3.1. *Let s be a partial k-shape, let $j \in [2k-4]$ such that every column of s is at least $\lceil (j+2)/2 \rceil$ cells high, and let $z \in \{0, 1, \dots, k-1-\lceil j/2 \rceil\}$. We consider two consecutive columns (from left to right) of s, which we denote by C_1 and C_2, with the same height and the same label but not the same level, and such that C_1 has been lifted in the context* (1) *of Definition 3.2 (note that it cannot be in the context* (2)*). If C_2 has been lifted at the same level as C_1 in $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$, then it is not in the context* (1) *of Definition 3.2.*

Proof. Let R_1 (resp. R_2) be the row in which C_1 (resp. C_2) is rooted, and let R be the row beneath R_1. Let l be the length of R. Since C_1 and C_2 have the same height and the same label, and since C_1 has been lifted in the context (1) of Definition 3.2, then it is necessary that the length of R_2 equals l as well. Consequently, the partial k-shape s is like depicted in Figure 11. We also consider the last column C_3 to be rooted in R_2, and the column C_4 on the right of C_3.

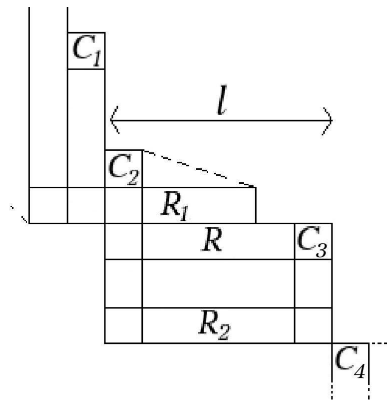

FIGURE 11. Partial k-shape s.

Now, suppose that, in $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$, the column C_2 has been lifed at the same level as C_1 in the context (1) of Definition 3.2. To do so, it is necessary that the row R gains cells between s and $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$, *i.e.*, that the column C_4 is lifted at the same level as C_3. By hypothesis, it means that C_4 must be lifted down to at least $\lceil (j+2)/2 \rceil$ cells between s and $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$. Obviously, every column of $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$ has been lifted up to at most $\lceil (j+1)/2 \rceil$ cells, so $\lceil (j+1)/2 \rceil = \lceil (j+2)/2 \rceil$, *i.e.*, there exists $p \in [k-2]$ such that $j = 2p$. Consequently, the partial k-shape $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$ is obtained by adding z columns of height $p+1$ and label 1 to s. However, according to the rule (3) of Definition 3.2, only the p top cells of those z columns may lift the columns of s in the context (1), *i.e.*, the columns of s are lifted up to at most p cells in this context, thence C_4 cannot be lifted at the same level as C_3, which is absurd. $\square$

Remark 3.2. Here, we give precisions about context (3) of Definition 3.2. Using the same notations, consider the column C_1 of s which contains the last cell (from left to right) of R_0, and C_2 the column which follows C_1 (see Figure 12). Since C_0 loses (momentarily) its saturation during the computation of $s \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^z$, it is necessary that the columns C_1 and C_2 have the same height and the same label in order to obtain the situation depicted in Figure 13. Consequently, the lifting of C_1 in Figure 12 comes from rule (1) of Definition 3.2 (it cannot be prompted by rule (3) because C_0 has not lost its saturation yet). Also, if the label of C_1 and C_2 is 1, then the hook length h_2 of the bottom cell c_2 of C_2 equals k in Figure 13

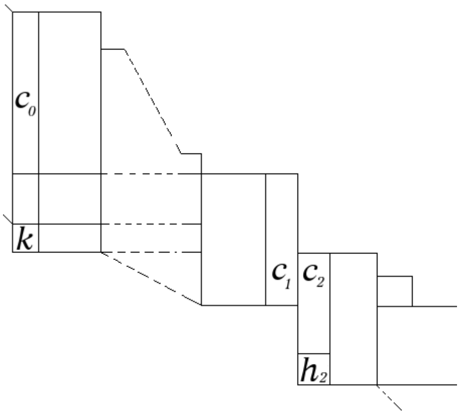

FIGURE 12. Partial k-shape s.

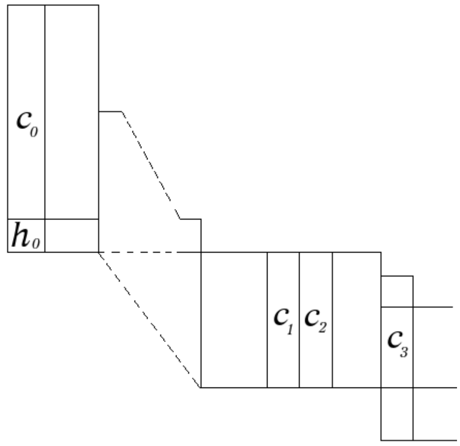

FIGURE 13. Between s and $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$.

(because C_1 has been lifted in the context (1)), implying the situation depicted in Figure 13 cannot be reached because, as noticed in Remark 3.1, the hook length h_2 of c_2 still equals k in Figure 13, forcing C_1 to be lifted by the rule (1). So, the label of C_1 and C_2 must be 2, and $h_2 = k - 1$.

Finally, according to Lemma 3.1, the lifting of C_2 in Figure 13 must be done in the context (2) of Definition 3.2: indeed, if it was context (3), there would exist a column C_0' labeled by 1 between C_0 and C_2, which would be lifted so that its bottom cell ends up in the row R_0 (in order for C_2 to be lifted at the same level). But then, since C_0 has not lost its saturation yet at this time (because C_2 has not been lifted yet), the column C_0' would be rooted in R_0 thus would be saturated, which is absurd because by hypothesis C_0' is supposed to lose momentarily its saturation. Thus, the column C_3 depicted in Figure 13 is labeled by 1.

Lemma 3.2. *Let s be a partial k-shape and $j \geq 1$ such that the height of every column of s is at least $\lceil (j+2)/2 \rceil$, and such that the quantity of integers $i \in [k-2]$ in which s is not saturated is at most $\lceil j/2 \rceil$. Then, if s is not saturated in $i_0 \in [k-2]$, there exists a unique integer $z \in [k - 1 - \lceil j/2 \rceil]$ such that the partial k-shape $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$ is saturated in i_0.*

Proof. According to Remark 3.2, columns labeled by 1 cannot be lifted in the context (3) of Definition 3.2. Consequently, in the partial k-shape s, the columns of height $i_0 + 1$ and label 1 are organized in $m \geq 1$ groups of columns rooted in a same row, such that the $m - 1$ first groups from right to left are made of saturated columns, *i.e.*, such that the columns of these groups are rooted in rows whose top left cell is hook lengthed by k, and such that the m-th group is made of non-saturated columns, *i.e.*, such that the columns $C_1, C_2, \dots, C_q$ of this group (from left to right) are rooted in a row whose top left cell (which is the bottom cell c_1 of C_1) is hook lengthed by some integer $h < k$ (see Figure 14).

Now, for all $p \in [k - \lceil j/2 \rceil]$, let s^p be the partial k-shape $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^p$. For all $p \geq 2$, the partial k-shape s^p is obtained by gluing a column of height $\lceil (j+1)/2 \rceil$ and label $t(j)$ right next to the last column of s^{p-1}, then by applying the 3 rules of Definition 3.2 so as to obtain a partial k-shape. We now focus on $s^{k - \lceil j/2 \rceil}$. Let C be a column of s. Since the height of C is at least $\lceil (j+2)/2 \rceil$, if the bottom cell c of C is located in the same row as one of the cells of the rectangle $\lceil (j+1)/2 \rceil^{k - \lceil j/2 \rceil}$ during the computation of $s^{k - \lceil j/2 \rceil}$, then the hook length of c will be at least $\lceil (j+2)/2 \rceil + k - \lceil j/2 \rceil \geq k+1$. Consequently, according to the rule (1) of Definition 3.2, the column C is lifted as long as its bottom cell is in the same row as one of the cell of the rectangle $\lceil (j+1)/2 \rceil^{k - \lceil j/2 \rceil}$, and since this holds for every column C of s, then the partial k-shape $s^{k - \lceil j/2 \rceil}$ is obtained by drawing the rectangle $\lceil (j+1)/2 \rceil^{k - \lceil j/2 \rceil}$ in the bottom right

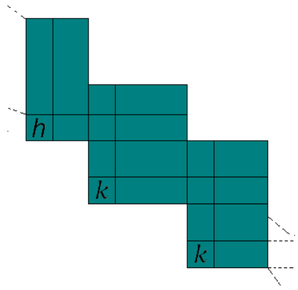

FIGURE 14. Partial k-shape s.

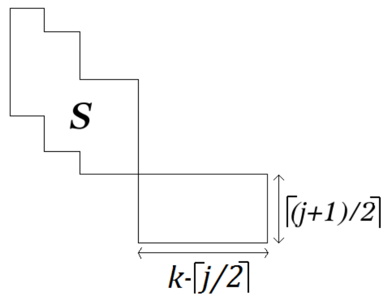

FIGURE 15. Partial k-shape $s^{k-\lceil j/2 \rceil}$.

hand corner of s (see Figure 15). In particular, the columns $C_1, C_2, \ldots, C_q$ must have been been lifted $\lceil (j+1)/2 \rceil$ times between s and $s^{k-\lceil j/2 \rceil}$. The idea is to prove there exists a unique $p_0 \in [k-1-\lceil j/2 \rceil]$ such that C_1 has been lifted in the context (1) of Definition 3.2 in s^{p_0+1}, implying the hook length h of c_1 equals k in s^{p_0}, which means s^{p_0} is saturated in i_0 according to Remark 3.1. Note that the columns $C_1, C_2, \ldots, C_q$ cannot be lifted in the context (3) of Definition 3.2 because their label is 1 (see Remark 3.2). If $m \geq 2$, the columns $C_1, C_2, \ldots, C_q$ cannot be lifted in the context (2), hence they are lifted in the context (1), so the existence and unicity of the integer p_0 is obvious. If $m = 1$, suppose C_1 is never lifted in the context (1), *i.e.*, that it is lifted $\lceil (j+1)/2 \rceil$ times in the context (2) between s and $s^{k-\lceil j/2 \rceil}$. Each of these $\lceil (j+1)/2 \rceil$ times, the first column labeled by 1 (from right to left) prompting the chain reaction of liftings in the context (2) (which leads to the lifting of C_1), must be different from the others. Also, the columns labeled by 1 responsible for these liftings cannot be saturated, because from Remark 3.1 they would still be saturated when glued to the columns of different height or label that are lifted, meaning their bottom cell would be hooked lengthed by k and that the lifting would be in the context (1), which is absurd. As a conclusion, it is necessary that in addition to i_0, there would exist $\lceil (j+1)/2 \rceil \geq \lceil j/2 \rceil$ different integers $i < i_0$ such that s is not saturated in i, which is absurd by hypothesis. □

4. Proof of Theorem 2.2

We first construct two key algorithms in the first two subsections.

4.1. Algorithm $\varphi : SP_{k-1} \to IS_k$.

Definition 4.1 (Algorithm φ)**.** Let $f \in SP_{k-1}$. We define $s^{2k-3}(f)$ as the empty skew partition. For j from $2k-4$ downto 1, let $i \in [k-1]$ such that $f(j) = 2i$, and suppose that the hypothesis $H(j+1)$ defined as "if $s^{j+1}(f)$ is not empty, the height of every column of $s^{j+1}(f)$ is at least $\lceil (j+2)/2 \rceil$, and the number of integers i in which $s^{j+1}(f)$ is not saturated is at most $\lceil j/2 \rceil$" is true (in particular $H(2k-3)$ is true so we can initiate the algorithm).

(1) If $f(2i) > 2i$, if $j = \min\{j' \in [2k-4], f(j') = 2i\}$ and if the partial k-shape s^{j+1} is not saturated in i, then we define $s^j(f)$ as $s^{j+1}(f) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(f)}$ where $z_j(f)$ is the unique element of $[k-1-\lceil j/2 \rceil]$ such that $s^{j+1}(f) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(f)}$ is saturated in i (see Lemma 3.2 in view of Hypothesis $H(j+1)$).

(2) Else, we define $s^j(f)$ as $s^{j+1}(f) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(f)}$ where $f(j) = 2(\lceil j/2 \rceil + z_j(f))$ (notice that $z_j(f) \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}$ by definition of a surjective pistol).

In either case, if $s^j(f)$ is not empty, then the height of every column is at least $\lceil (j+1)/2 \rceil$. Also, suppose there exists at least $\lceil (j-1)/2 \rceil + 1$ different integers $i \in [k-2]$ in which $s^j(f)$ is not saturated. In view of the rule (1) of the present algorithm, this implies there are at least $\lceil (j-1)/2 \rceil + 1$ integers $j' \leq j-1$ such that $f(j') \geq 2\lceil j/2 \rceil$. Also, since f is surjective, there exist at least $\lceil j/2 \rceil - 1$ integers $j'' \leq j-1$ such that $f(j') \leq 2(\lceil j/2 \rceil - 1)$. Consequently, we obtain $(\lceil (j-1)/2\rceil + 1) + (\lceil j/2\rceil - 1) \leq j-1$, which cannot be because $\lceil (j-1)/2\rceil + \lceil j/2\rceil = j$. So the hypothesis $H(j)$ is true and the algorithm goes on. Ultimately, we define $\varphi(f)$ as the partition $< s^1(f) >$.

Proposition 4.1. *For all $f \in SP_{k-1}$, the partition $\lambda = \varphi(f)$ is an irreducible k-shape such that $\partial^k(\lambda) = s^1(f)$ and $\overrightarrow{fr}(\lambda) = \overrightarrow{fix}(f)$.*

For example, consider the surjective pistol $f = (2, 8, 4, 10, 10, 6, 8, 10, 10, 10) \in SP_5$ whose tableau is depicted in Figure 16. Apart from 10, the only fixed point of f is 6, so $\overrightarrow{fix}(f) = (0, 0, 1, 0)$.

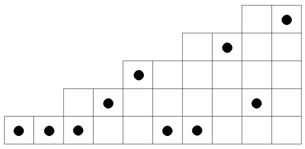

FIGURE 16. Surjective pistol $f = (2, 8, 4, 10, 10, 6, 8, 10, 10, 10) \in SP_5$.

Algorithm φ provides the sequence $(s^8(f), s^7(f), \ldots, s^1(f))$ depicted in Figure 17 (note that $s^8(f) = s^7(f) = s^6(f)$ because $z_7(f) = z_6(f) = 0$).

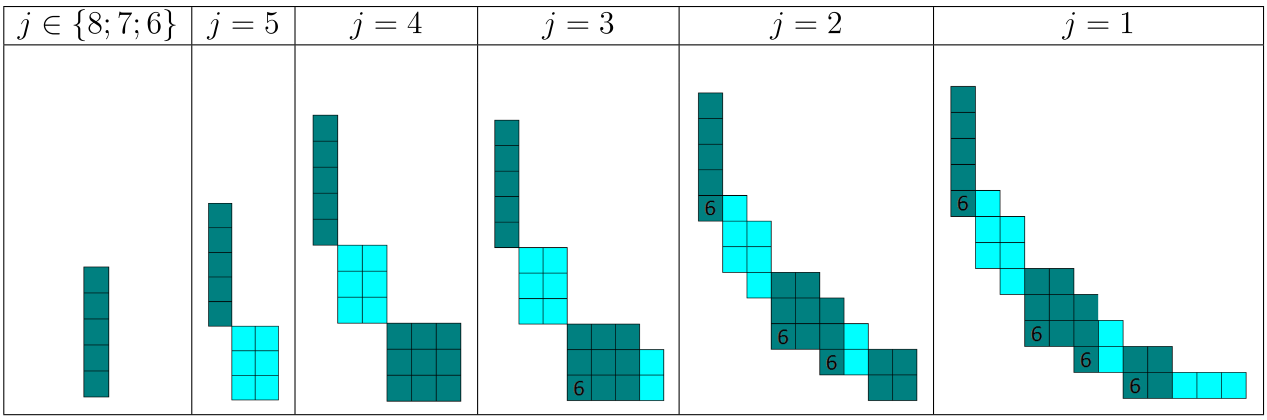

FIGURE 17. Sequence $(s^j(f))_{j\in[8]}$.

Thus, we obtain $s^1(f) = \partial^6(\lambda)$ where $\lambda = \varphi(f) =< s^1(f) >$. In particular, the sequences $rs^6(\lambda) = (5, 4, 4, 3, \ldots, 1)$ and $cs^6(\lambda) = (5, 3, 3, 3, \ldots, 1)$ are partitions, so λ is a 6-shape. Finally, we can see in Figure 18 that λ is irreducible and $\overrightarrow{fr}(\lambda) = (0, 0, 1, 0) = \overrightarrow{fix}(f)$.

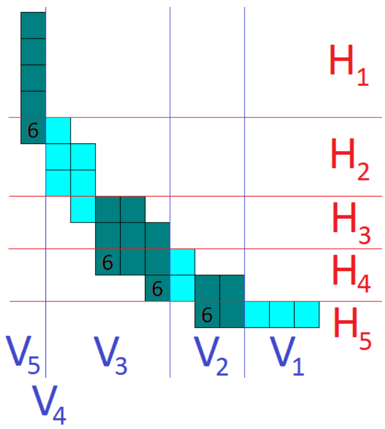

FIGURE 18. 6-boundary $\partial^6(\lambda) = s^1(f)$ of the irreducible 6-shape $\lambda = \varphi(f)$.

We split the proof of Proposition 4.1 into Lemmas 4.1, 4.2 and 4.3.

Lemma 4.1. *For all $f \in SP_{k-1}$, we have $\partial^k(\varphi(f)) = s^1(f)$.*

Proof. By construction, the skew partition $s^1(f)$ is a saturated partial k-shape (the saturation is guaranteed by Hypothesis $H(1)$). As a partial k-shape, the hook length of every cell doesn't exceed k. Consequently, to prove that $\partial^k(\varphi(f)) = s^1(f)$, we only need to show that the hook length h_1 of every *anticoin* of $s^1(f)$ (namely, cells glued simultaneously to the left of a row of $s^1(f)$ and beneath a column of $s^1(f)$, see Figure 19) is such that $h_1 > k$.

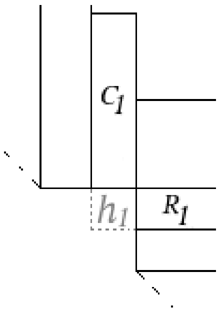

FIGURE 19. Anticoin labeled by its hook length h_1.

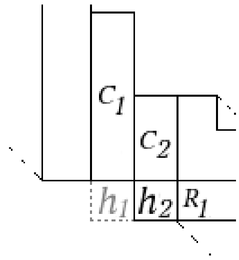

FIGURE 20

Anticoins of $s^1(f)$ are created by lifting columns in one of the three contexts (1),(2) or (3) of Definition 3.2. Let x_1 (resp. y_1) be the length of the row R_1 (resp. the height of the column C_1).

(1) If C_1 has been lifted in the context (1), then $x_1 + y_1 > k$ (if c_1 is labeled by 1) or $x_1 + y_1 > k - 1$ (if C_2 is labeled by 2). In either case, we obtain $h_1 = 1 + x_1 + y_1 > k$.
(2) If C_1 has been lifed in the context (2), then the first cell (from left to right) of the row R_1 is a corner, and it is the bottom cell of a column C_2 labeled by 1 (see Figure 20). Let y_2 be the height of C_2. Since C_2 is saturated, then the hook length $h_2 = x_1 + y_2 - 1$ of its bottom cell equals k. Consequently, since $y_1 \geq y_2$, we obtain $h_1 = x_1 + y_1 - 1 > k$.
(3) Else C_1 has been lifted in the context (3). Let C_0 be the saturated column of $s^1(f)$ such that C_1 is the column that contains the last cell (from left to right) of the row R_0 in which C_0 is rooted (see Figure 21).

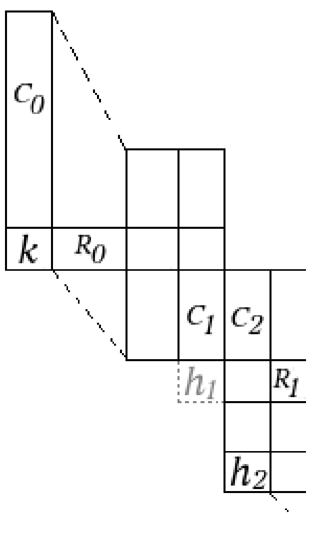

FIGURE 21. $s^1(f)$.

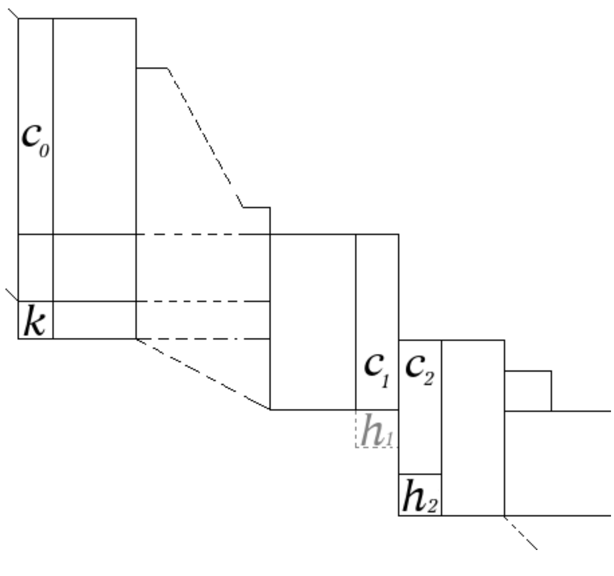

FIGURE 22. $s^{j+1}(f)$.

Let $j \in [2k-4]$ such that C_0 is saturated in the partial k-shape $s^{j+1}(f)$ and such that

C_0 loses its saturation at some point of the computation of $s^j(f)$ (recall that C_0 is saturated at the end of this computation by the rule (3) of Definition 3.2). The partial k-shape $s^{j+1}(f)$ then presents the situation depicted in Figure 22. Following Remark 3.2, the columns C_1 and C_2 have the same height and the same label 2 (and $h_2 = k-1$), and in order for C_0 to lose temporarily its saturation between $s^{j+1}(f)$ and $s^j(f)$, there exists a column C_3 labeled by 1 on the right of C_2 such that C_2 is lifted at the same level as C_1 in the context (2) of Definition 3.2, with C_3 being the column responsible for this lifting (see Figure 23).

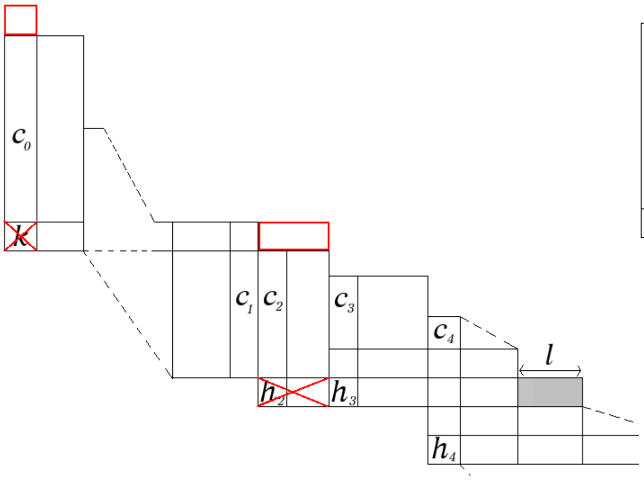

FIGURE 23. Between $s^{j+1}(f)$ and $s^j(f)$.

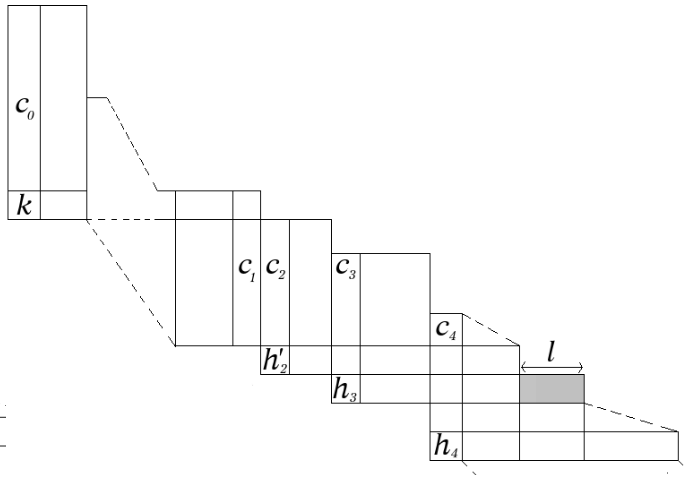

FIGURE 24. $s^j(f)$

Now, the rule (3) of Definition 3.2 yields the situation depicted in Figure 24, in which $h_2' = h_2 - l$ where $l \geq 1$ is the number of gray cells.
Finally, in order to saturate C_3 (every column labeled by 1 being saturated in $s^1(f)$), it is necessary to lift the l gray cells. Indeed, otherwise, the column C_3 would become saturated by lifting columns $C_5, C_6, \ldots, C_m$ whose top cells would be glued to the right of the last gray cells, meaning those columns have the same height y and the same label t as the l columns whose top cells are the gray cells. Obviously, since C_3 is not saturated yet at that moment, the columns $C_5, C_6, \ldots, C_m$ are not lifted in the context (3) of Definition 3.2. In view of Lemma 3.1, these columns must have been lifted in the context (2), which implies every column of height y and label t are lifted to the same level as the last gray cell. But then it would lift the column C_3 in the context (1) of Definition 3.2 instead of saturating it, which is absurd. So the l gray cells are necessarily lifted in $s^1(f)$ (see Figure 25), in which h_2' has become h_2 again).
In particular, the hook length $h_2' = h_2$ equals $k-1$ and we obtain $h_1 = h_2 + 2 > k$. □

Lemma 4.2. *For all $f \in SP_{k-1}$, the partition $\lambda = \varphi(f)$ is a k-shape.*

Proof. From Lemma 4.1 we know that $s^1(f) = \partial^k(\lambda)$, and since $s^1(f)$ is a partial k-shape by construction, the sequence $cs^k(\lambda) = cs(s^1(f))$ is a partition. To prove that λ is a k-shape, it remains to show that the sequence $rs^k(\lambda) = rs(s^1(f))$ is a partition. Let R and R' be two consecutive rows (from bottom to top) of $s^1(f)$ (see Figure 26). Let x, x' be the respective lengths of R, R' and $y_1 \geq y_2 \geq y_3$ the respective heights of the columns C_1, C_2, C_3 introduced by the picture. Since $s^1(f) = \partial^k(\lambda)$, we know that the quantity $h = y_2 + x + 1$ (which is the hook length of the cell glued to C_2 and R) exceeds k. Also, the hook length h_1 of the bottom cell c_1 of C_1 doesn't exceed k, so $x' = h_1 - y_1 + 1 \leq k - y_1 + 1 \leq k - y_2 + 1 \leq x + 1$. Now suppose that $x' = x + 1$. Then $h_1 = y_1 + x' - 1 = y_1 + x \geq y_2 + x = h - 1 \geq k$, so $h_1 = k$ (which implies $y_1 = y_2$). Consequently C_1 and C_2 are two columns of height $y_1 = y_2$, and labeled by 1 because

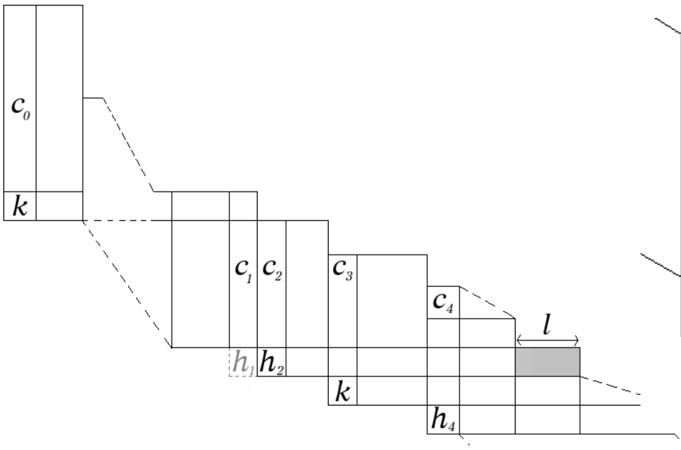

FIGURE 25. $s^1(f)$

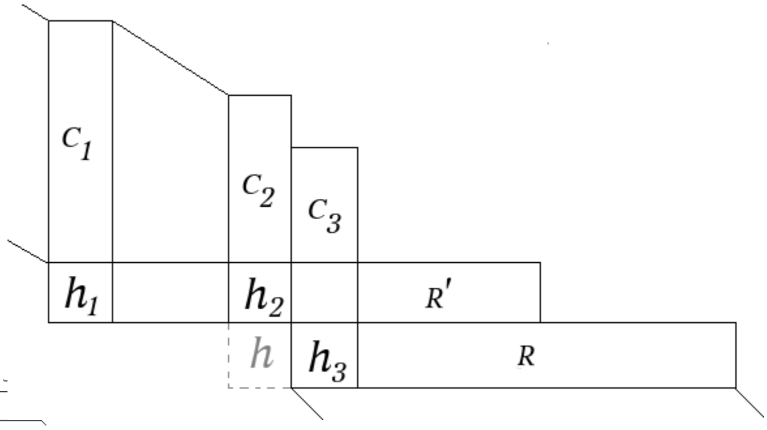

FIGURE 26. s^1

$h_1 = k$. Also, we have $h_3 = y_3 + x - 1 = y_3 + x' - 2 \leq y_1 + x' - 2 = k - 1$. Since every column labeled by 1 in $s^1(f)$ is saturated, it forces C_3 to be labeled by 2, which implies the column C_2 has not been lifted in the context (2) of Definition 3.2. Now $y_2 + x = y_1 + x' - 1 = k$ so C_2 has not been lifted in the context (1) of Definition 3.2 either. Consequently C_2 has been lifted in the context (3). Now, following Remark 3.2, it implies C_2 and C_3 have the same height, *i.e.*, that $y_2 = y_3$. This is impossible because, by construction of $\varphi(f)$, columns of height $y_2 = y_3$ labeled by 2 (including C_3) are positionned before columns of height y_2 labeled by 1 (including C_2). As a conclusion, it is necessary that $x \geq x'$, then $rs^k(\lambda)$ is a partition and λ is a k-shape. □

Lemma 4.3. *For all $f \in SP_{k-1}$, the k-shape $\lambda = \varphi(f)$ is irreducible and $\overrightarrow{fr}(\lambda) = \overrightarrow{fix}(f)$.*

Proof. For all $i \in [k-2]$, let n_i (resp. m_i) be the number of horizontal steps of the k-rim of λ that appear inside the set $H_{k-i}(\lambda) \cap V_{i+1}(\lambda)$ (resp. inside the set $H_{k-i}(\lambda) \cap V_i(\lambda)$). Recall that λ is irreducible if and only if $(n_i, m_i) \in \{0, 1, \ldots, k-1-i\}^2$ for all $i \in [k-2]$. Consider $i_0 \in [k-2]$. The number n_{i_0} is precisely the number of saturated columns of height $i_0 + 1$ of the partial k-shape $s^1(f) = \partial^k(\lambda)$. Since $s^1(f)$ is saturated by construction, this number is the quantity $z_{2i_0}(f) < k - i_0$ according to Definition 4.1. This statement being true for any $i_0 \in [k-2]$, in particular, if $i_0 > 1$, there are $n_{i_0-1} = z_{2i_0-2}(f)$ columns of height i_0 and label 1 in $s^1(f)$, thence the quantity m_{i_0} is precisely the number $z_{2i_0-1}(f) < k - i_0$ of columns of height i_0 and label 2. Also, the columns of height 1 are necessarily labeled by 2, so $m_1 = z_1(f) < k-1$. Consequently, the k-shape λ is irreducible. Finally, for all $i \in [k-2]$, we have the equivalence $f(2i) = 2i \Leftrightarrow z_{2i}(f) = 0$. Indeed, if $f(2i) = 2i$ then by definition $z_{2i}(f) = f(2i)/2 - i = 0$. Reciprocally, if $f(2i) > 2i$, then either $z_{2i}(f)$ is defined in the context (1) of Definition 4.1, in which case $z_{2i}(f) > 0$, or $z_{2i}(f) = f(2i)/2 - i > 0$. Therefore, the equivalence is true and exactly translates into $\overrightarrow{fix}(f) = \overrightarrow{fr}(\lambda)$. □

4.2. **Algorithm $\phi : IS_k \to SP_{k-1}$.**

Definition 4.2. Let λ be an irreducible k-shape. For all $i \in [k-2]$, we denote by $x_i(\lambda)$ the number of horizontal steps of the k-rim of λ inside the set $H_{k-i}(\lambda) \cap V_{i+1}(\lambda)$, and by $y_i(\lambda)$ the number of horizontal steps inside the set $V_i(\lambda) \backslash H_{k+1-i}(\lambda) \cap V_i(\lambda) = \bigsqcup_{j=1}^{k-i} H_j(\lambda) \cap V_i(\lambda)$. Finally, for all $j \in [2k-4]$, we set

$$z_j(\lambda) = \begin{cases} y_i(\lambda) & \text{if } j = 2i-1, \\ x_i(\lambda) & \text{if } j = 2i. \end{cases}$$

For example, if λ is the irreducible 6-shape represented in Figure 18, then $(z_j(\lambda))_{j\in[8]} = (3,2,1,3,2,0,0,1)$. Note that in general, if λ is an irreducible k-shape and $(t_1, t_2, \ldots, t_{k-2}) = \overrightarrow{fr}(\lambda)$, then $t_i = 1$ if and only if $x_i(\lambda) = 0$, for all i.

Lemma 4.4. *For all $\lambda \in IS_k$ and for all $j \in [2k-4]$, we have*

$$z_j(\lambda) \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}.$$

Proof. By definition of an irreducible k-shape, we automatically have $z_{2i}(\lambda) = x_i(\lambda) < k-i$ for all $i \in [k-2]$. The proof of $z_{2i-1}(\lambda) = y_i(\lambda) < k-i$ is less straightforward. Suppose that $y_i(\lambda) \geq k-i$. Let C_0 be the first column (from left to right) of $\bigsqcup_{j=1}^{k-i} H_j(\lambda) \cap V_i(\lambda)$, let R_0 the row in which C_0 is rooted, and let R_1 be the row beneath R_1. We denote by $l \in [y_i(\lambda)]$ the number of consecutive columns of $\bigsqcup_{j=1}^{k-i} H_j(\lambda) \cap V_i(\lambda)$ whose bottom cells are located R_0, and l' the length of R_1 (see Figure 27).

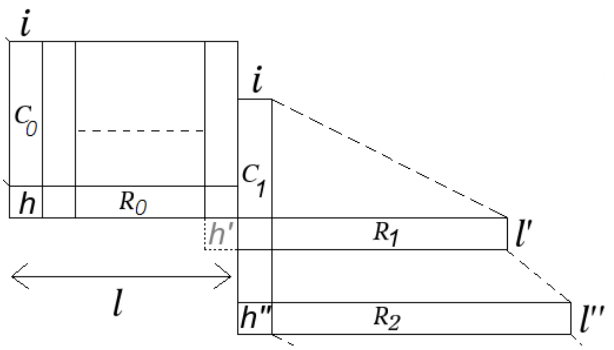

FIGURE 27

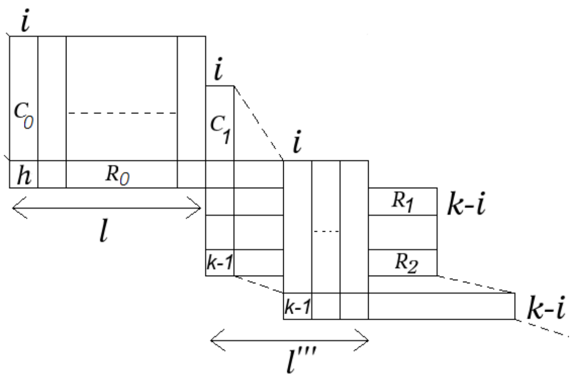

FIGURE 28

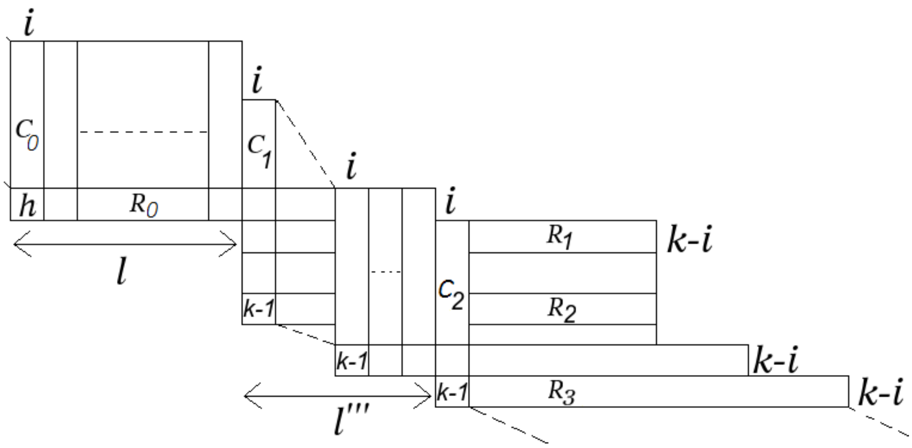

FIGURE 29

The hook cell $h' = i + l' + 1$ of the cell glued to the left of R_1 exceeds k, thence $l' \geq k-i$. Now suppose that $l \geq k-i$: then, the hook length $h \leq k-1$ of the bottom cell c_0 of C_0 is such that $h \geq i + l - 1 \geq k-1$ hence $h = k-1$. As a result, the first l columns of $\bigsqcup_{j=1}^{k-i} H_j(\lambda) \cap V_i(\lambda)$ are in fact located in $H_{k-i}(\lambda) \cap V_i(\lambda)$, therefore $|H_{k-i}(\lambda) \cap V_i(\lambda)| \geq l \geq k-i$, which contradicts the irreducibility of λ. So $l \leq k-i-1 < y_i(\lambda)$. As a consequence, there exists a column of $\bigsqcup_{j=1}^{k-i} H_j(\lambda) \cap V_i(\lambda)$ which intersects R_1; in particular, consider the first column C_1 that does so, then its bottom cell c_1 is located in a row R_2 (whose length is denoted by $l'' \geq l'$) and c_1 is hooked lengthed by $h'' = i + l'' - 1 < k$, thence $k-i \leq l' \leq l'' \leq k-i$, which implies $l' = l'' = k-i$. Now, let $l''' \in \{0, 1, \ldots, k-i\}$ be the number of columns of $\bigsqcup_{j=1}^{k-i} H_j(\lambda) \cap V_i(\lambda)$ intersecting R_1 but whose top cells are not located in R_1 (see Figure 28).

Since $h = i-1+l+l''' \leq k-1$, we obtain $l+l''' \leq k-i$. With precision, we must have $l+l''' \leq k-i-1$: otherwise, we would have $l+l''' = k-i$ and $h = k-1$ which implies the first $l+l''' = k-i$ columns of $\bigsqcup_{j=1}^{k-i} H_j(\lambda)\cap V_i(\lambda)$ are located in $H_{k-i}(\lambda)\cap V_i(\lambda)$, which cannot be because λ is irreducible. So $l+l''' < k-i \leq y_i(\lambda)$. It means there exists a column C_2 of $\bigsqcup_{j=1}^{k-i} H_j(\lambda)\cap V_i(\lambda)$ whose top box is located in R_1, which forces its bottom cell to be located in a row R_3 of length $k-i$ (see Figure 29) because $rs(\lambda)$ is a partition. But then the bottom cell of every column intersecting R_1 is located in a row of length $k-i$, therefore the bottom cells of those $k-i$ columns are elements of the set $H_{k-i}(\lambda)\cap V_i(\lambda)$, which cannot be because λ is irreducible and the length of R_1 is $k-i$. As a conclusion, it is necessary that $y_i(\lambda) < k-i$. $\square$

Definition 4.3. Let $\lambda \in IS_k$. We define a sequence $(s^j(\lambda))_{j\in[2k-3]}$ of partial k-shapes by $s^{2k-3}(\lambda) = \varnothing$ and

$$s^j(\lambda) = s^{j+1}(\lambda) \oplus_{t(j)}^k \lceil (j+1)/2 \rceil^{z_j(\lambda)}.$$

Lemma 4.5. *We have $s^1(\lambda) = \partial^k(\lambda)$ for all $\lambda \in IS_k$.*

Proof. Let n be the number of columns of $[\lambda]$, which is obviously the same as for $\partial^k(\lambda)$ and $s^1(\lambda)$. For all $q \in [n]$, we define $\partial^k(\lambda)_q$ (respectively $s^1(\lambda)_q$) as the skew partition (resp. labeled skew partition) obtained by considering the q first columns (from right to left) of $\partial^k(\lambda)$ (resp. $s^1(\lambda)$). The idea is to prove that $\partial^k(\lambda)_q = s^1(\lambda)_q$ for all $q \in [n]$ by induction (the statement being obvious for $q = 1$). In particular, for $q = n$, we will obtain $\partial^k(\lambda) = s^1(\lambda)$. Suppose that $\partial^k(\lambda)_q = s^1(\lambda)_q$ for some $q \geq 1$. The $(q+1)$-th column C_{q+1} (whose bottom cell is denoted by c_{q+1}) of $\partial^k(\lambda)_{q+1}$ (from right to left) is glued to the left of $\partial^k(\lambda)_q$, at the unique level such that the hook length h of c_{q+1} doesn't exceed k, and the hook length x of the cell beneath c_{q+1} exceeds k. Since the hook length of every cell of $s^1(\lambda)$ doesn't exceed k, the $(q+1)$-th column C'_{q+1} (whose bottom cell is denoted by c'_{q+1}) of $s^1(\lambda)_{q+1}$ is necessarily positionned above or at the same level as C_{q+1} (see Figure 30 where C_{q+1} [resp. C'_{q+1}] has been drawed in black [resp. in red] and whose bottom cell is labeled by its hook length h [resp. h']).

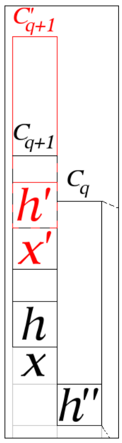

FIGURE 30. Skew partitions $\partial^k(\lambda)_{q+1}$ and $s^1(\lambda)_{q+1}$.

Now, suppose that the columns C_{q+1} and C'_{q+1} are not at the same level. In particular, the hook length x' of the cell beneath c'_{q+1} is such that $x' \leq h \leq k$. Also, the bottom cell c'_q of the q-th column C'_q of $s^1(\lambda)_{q+1}$ is a corner.

(1) If C'_{q+1} has been lifted in the context (1) of Definition 3.2, then since $x' \leq k$, the label of C'_{q+1} is necessarily 2, and $x' = k$. Consequently, the cell c_{q+1} is in fact the cell labeled by x', and $h = x' = k$. In particular, this implies C'_{q+1} is labeled by 1, which is absurd because C_{q+1} and C'_{q+1} must have the same label.

(2) If C'_{q+1} has been lifted in the context (2) of Definition 3.2, then in particular C'_q is a column labeled by 1. Consequently, the hook length of c'_q, which is the same as c_q because $s^1(\lambda)_q = \partial^k(\lambda)_q$, is the integer $h'' = k$. Since $rs^k(\lambda)$ and $cs^k(\lambda)$ are partitions, this implies $h > h'' = k$, which is absurd.

(3) Therefore, the column C'_{q+1} has necessarily been lifted in the context (3) of Definition 3.2. According to Remark 3.2, it implies :
   (a) C'_{q+1} and C'_q have the same height and the same label 2 (and $h'' = k-1$);
   (b) C'_{q+1} is located one cell higher than C'_q.

   In particular, from (b), since C'_{q+1} is supposed to be located at a higher level than C_{q+1}, then the cell c_{q+1} is glued to the left of the cell c_q. Since $h'' = k-1$, we obtain $h = k$, which is in contradiction with C'_{q+1} being labeled by 2.

So C_{q+1} and C'_{q+1} are located at the same level, thence $\partial^k(\lambda)_{q+1} = s^1(\lambda)_{q+1}$. □

Notice that Lemma 4.5 is obvious if we know that $\lambda = \varphi(f)$ for some surjective pistol $f \in SP_{k-1}$, because in that case $s^j(\lambda) = s^j(f)$ for all j.

Definition 4.4 (Algorithm ϕ)**.** Let $\lambda \in IS_k$. We define $m(\lambda) \in \{0, 1, \ldots, k-2\}$ and

$$1 \leq i_1(\lambda) < i_2(\lambda) < \ldots i_m(\lambda) \leq k-2$$

such that

$$\{i_1(\lambda), i_2(\lambda), \ldots, i_{m(\lambda)}(\lambda)\} = \{i \in [k-2], x_i(\lambda) > 0\}$$

(this set may be empty). For all $p \in [m(\lambda)]$, let

$$j_p(\lambda) = \max\{j \in [2i_p(\lambda)-1], s^j(\lambda) \text{ is saturated in } i_p(\lambda)\}.$$

Let $L(\lambda) = [2k-4]$. For j from 1 to $2k-4$, if $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$, and if there is no $j' \in L(\lambda)$ such that $j' < j$ and $\lceil j'/2 \rceil + z_{j'} = i_p(\lambda)$, then we set $L(\lambda) := L(\lambda) \backslash \{j_p(\lambda)\}$. Now we define $\phi(\lambda) \in \mathbb{N}^{[2k-2]}$ as the following: the integers $\phi(\lambda)(2k-2)$ and $\phi(\lambda)(2k-3)$ are defined as $2k-2$; afterwards, let $j \in [2k-4]$.

- If $j \in L(\lambda)$ then $\phi(\lambda)(j)$ is defined as $2(\lceil j/2 \rceil + z_j(\lambda))$.
- Else there exists a unique $p \in [m(\lambda)]$ such that $j = j_p(\lambda)$, and we define $\phi(\lambda)(j)$ as $2i_p(\lambda)$.

Proposition 4.2. *For all $\lambda \in IS_k$, the map $\phi(\lambda)$ is a surjective pistol of height $k-1$, such that $\overrightarrow{fix}(\phi(\lambda)) = \overrightarrow{fr}(\lambda)$.*

For example, consider the irreducible 6-shape λ of Figure 18, such that $(z_j(\lambda))_{j\in[8]} = (3,2,1,3,2,0,0,1)$. In particular $(x_1(\lambda), x_2(\lambda), x_3(\lambda), x_4(\lambda)) = (2,3,0,1)$ so $m(\lambda) = 3$ and $(i_1(\lambda), i_2(\lambda), i_3(\lambda)) = (1,2,4)$. Moreover, by considering the sequence of partial 6-shapes $(s^8(\lambda), \ldots, s^1(\lambda))$, which is in fact the sequence $(s^8(f), \ldots, s^1(f))$ depicted in Figure 17 (with $f = (2,8,4,10,10,6,8,10,10,10) \in SP_5$) because $\lambda = \varphi(f)$, we obtain $(j_2(\lambda), j_3(\lambda), j_1(\lambda)) = (3,2,1)$. Applying the algorithm of Definition 4.4 on $L(\lambda) = [8]$, we quickly obtain $L(\lambda) = \{4,5,6,7,8\}$. Consequently, if $g = \phi(\lambda)$, then automatically $g(10) = g(9) = 10$, afterwards $(g(1), g(2), g(3)) = (g(j_1(\lambda)), g(j_3(\lambda), g(j_2(\lambda)) = (2i_1(\lambda), 2i_3(\lambda), 2i_2(\lambda)) = (2,8,4)$ since $j_p(\lambda) \notin L(\lambda)$ for all $p \in [3]$, and $g(j) = 2(\lceil j/2 \rceil + z_j(\lambda))$ for all $j \in L(\lambda)$. Finally, we obtain $g = (2,8,4,10,10,6,8,10,10,10) = f$ (and $\overrightarrow{fix}(g) = \overrightarrow{fr}(\lambda)$).

Proof of Proposition 4.2. Let $\lambda \in IS_k$ and $f = \phi(\lambda)$. We know that $f(2k-2) = f(2k-3) = 2k-4$. Consider $j \in [2k-4]$.

(1) If $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$ and if $j \notin L(\lambda)$, then $f(j) = 2i_p(\lambda)$. By definition $2i_p(\lambda) > j_p(\lambda)$, so $2k-2 \geq f(j) > j$.
(2) Else $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda))$, so $2k-2 \geq f(j) \geq j$ following Lemma 4.4.

Consequently f is a map $[2k-2] \to \{2,4,\dots,2k-2\}$ such that $f(j) \geq j$ for all $j \in [2k-2]$. Now, we prove that f is surjective. We know that $2k-2 = f(2k-2)$. Let $i \in [k-2]$.

- If $i = i_p(\lambda)$ for some $p \in [m(\lambda)]$, then either $j_p(\lambda) \notin L(\lambda)$, in which case $2i = f(j_p(\lambda))$, or there exists $j < j_p(\lambda)$ in $L(\lambda)$ such that $\lceil j/2 \rceil + z_j = i$, in which case $2i = f(j)$.
- Else $z_{2i}(\lambda) = 0$, which implies that $2i$ cannot be equal to any $j_p(\lambda)$ because $s^{2i}(\lambda) = s^{2i+1}(\lambda) \oplus_1^k (i+1)^{z_j(\lambda)} = s^{2i+1}(\lambda)$. Consequently $2i \in L(\lambda)$, thence $f(2i) = 2(i + z_{2i}(\lambda)) = 2i$.

Therefore $f \in SP_{k-1}$. Finally, for all $i \in [k-2]$, we have just proved that $z_{2i}(\lambda) = 0$ implies $f(2i) = 2i$. Reciprocally, if $f(2i) = 2i$, then necessarily $2i \in L(\lambda)$ (otherwise $2i$ would be $j_p(\lambda)$ for some p and $f(2i)$ would be $2i_p(\lambda) > j_p(\lambda) = 2i$), meaning $2i = f(2i) = 2(i + z_{2i}(\lambda))$ thence $z_{2i}(\lambda) = 0$. The equivalence $z_{2i}(\lambda) = 0 \Leftrightarrow f(2i) = 2i$ for all $i \in [k-2]$ exactly translates into $\overrightarrow{fr}(\lambda) = \overrightarrow{fix}(f)$. $\square$

4.3. **Proof of Theorem 2.2.** At this stage, we know that φ is a map $SP_{k-1} \to IS_k$ that transforms the statistic $\overrightarrow{fix}$ into the statistic $\overrightarrow{fr}$. The bijectivity of φ is a consequence of the following proposition.

Proposition 4.3. *The maps $\varphi : SP_{k-1} \to IS_k$ and $\phi : IS_k \to SP_{k-1}$ are inverse maps.*

Lemma 4.6. *Let $(f, \lambda) \in SP_{k-1} \times IS_k$ such that $\lambda = \varphi(f)$ or $f = \phi(\lambda)$. Let $p \in [m(\lambda)]$ and $j^p(\lambda) := \min\{j \in [2k-4], f(j) = 2i_p(\lambda)\}$. The two following assertions are equivalent.*

1. *$j_p(\lambda) \notin L(\lambda)$.*
2. *$j_p(\lambda) = j^p(\lambda)$.*

Proof. Let $f \in SP_{k-1}$ and $\lambda = \varphi(f)$. In particular, we have $s^j(\lambda) = s^j(f)$ and $z_j(\lambda) = z_j(f)$ for all $j \in [2k-4]$. For all $p \in [m(\lambda)]$, by Definition 4.1 the partial k-shape $s^{j^p(\lambda)}(f) = s^{j^p(\lambda)}(\lambda)$ is necessarily saturated in $i_p(\lambda)$, thence $j_p(\lambda) \geq j^p(\lambda)$.

1. If $j_p(\lambda) \notin L(\lambda)$, suppose that $j_p(\lambda) > j^p(\lambda)$. Then, the partial k-shape $s^{j^p(\lambda)+1}(f) = s^{j^p(\lambda)+1}(\lambda)$ is saturated in $i_p(\lambda)$, meaning the integer $z_{j^p(\lambda)}(f) = z_{j^p(\lambda)}(\lambda)$ is defined as $f(j^p(\lambda))/2 - \lceil j^p(\lambda)/2 \rceil = i_p(\lambda) - \lceil j^p(\lambda)/2 \rceil$. Consequently, since $j_p(\lambda) \notin L(\lambda)$ and $j^p(\lambda) < j_p(\lambda)$, the integer $j^p(\lambda)$ cannot belong to $L(\lambda)$ either. So $j^p(\lambda) = j_{p_1}(\lambda)$ for some $p_1 \neq p$ because $j_p(\lambda) \neq j^p(\lambda)$. Also, since $f(j_{p_1}(\lambda)) = 2i_p(\lambda) \neq 2i_{p_1(\lambda)}$, then $j_{p_1}(\lambda) > j^{p_1}(\lambda)$ (and $j_{p_1}(\lambda) = j^p(\lambda) \notin L(\lambda)$). By iterating, we build an infinite decreasing sequence $(j^{p_i}(\lambda))_{i \geq 1}$ of distinct elements of $[2k-4]$, which is absurd. Therefore, it is necessary that $j_p(\lambda) = j^p(\lambda)$.
2. Reciprocally, if $j_p(\lambda) = j^p(\lambda)$, suppose that $j_p(\lambda) \in L(\lambda)$. Then, there exists $j \in L(\lambda)$ such that $j < j_p(\lambda)$ and $z_j(\lambda) = i_p(\lambda) - \lceil j/2 \rceil$. Let $i \in [k-1]$ such that $f(j) = 2i$ (since $j < j_p(\lambda) = j^p(\lambda)$, we know that $i \neq i_p(\lambda)$). Suppose $s^j(f)$ is defined in the context (1) of Definition 4.1. In particular $i = i_{p_1}(\lambda)$ for some $p_1 \in [m(\lambda)]$ (because $f(2i) > 2i$ which implies $z_{2i}(\lambda) = z_{2i}(f) > 0$), and $j = j^{p_1}(\lambda)$ and $s^{j+1}(f)$ is not saturated in i. Then, by definition the partial k-shape $s^j(f)$ is the first partial k-shape to be saturated in $i_{p_1}(\lambda)$ in the sequence $(s^{2k-4}(f) = s^{2k-4}(\lambda), \dots, s^1(f) = s^1(\lambda))$, meaning $j = j_{p_1}(\lambda)$. To sum up, the integer $j = j^{p_1}(\lambda) = j_{p_1}(\lambda)$ doesn't belong to $L(\lambda)$, and $p_1 \neq p$ because $f(j) = 2i_{p_1}(\lambda)$ and $j < j^p(\lambda) = j_p(\lambda)$. By iterating, we build an infinite decreasing sequence $(j^{p_i}(\lambda))_{i \geq 1}$ of elements of $[2k-4]$, which is absurd. So $s^{j+1}(f)$ is necessarily defined in the context (2) of Definition 4.1, meaning $z_j(f) = f(j)/2 - \lceil j/2 \rceil$). Since $z_j(f) = z_j(\lambda) = i_p(\lambda) - \lceil j/2 \rceil$, we obtain $f(j) = 2i_p(\lambda)$, which is in contradiction with $j_p(\lambda) = j^p(\lambda) > j$. As a conclusion, it is necessary that $j_p(\lambda) \notin L(\lambda)$.

Now let $\lambda \in IS_k$ and $f = \phi(\lambda)$. We consider $p \in [m(\lambda)]$.

1. If $j_p(\lambda) \notin L(\lambda)$, suppose that $j_p(\lambda) \neq j^p(\lambda)$. Then, by definition $f(j_p(\lambda)) = 2i_p(\lambda)$, meaning $j_p(\lambda) > j^p(\lambda)$. Suppose now that $j^p(\lambda) \in L(\lambda)$, then $2i_p(\lambda) = f(j^p(\lambda)) =$

$2(\lceil j^p(\lambda)/2\rceil + z_{j^p(\lambda)}(\lambda))$. As a result, we obtain $z_{j^p(\lambda)}(\lambda) = i_p(\lambda) - \lceil j^p(\lambda)/2\rceil$, which is in contradiction with $j_p(\lambda) \notin L(\lambda)$. So $j^p(\lambda) \notin L(\lambda)$, which implies $j^p(\lambda) = j_{p_1}(\lambda)$ for some $p_1 \neq p$, and necessarily $j_{p_1}(\lambda) \neq j^{p_1}(\lambda)$ since $f(j_{p_1}(\lambda)) = 2i_p(\lambda) \neq 2i_{p_1}(\lambda)$. By iterating, we build a sequence $(j^{p_i}(\lambda))_{i\geq 1}$ of distinct elements of $[2k-4]$, which is absurd. So $j_p(\lambda) = j^p(\lambda)$.

(2) Reciprocally, if $j_p(\lambda) = j^p(\lambda)$, suppose that $j_p(\lambda) \in L(\lambda)$. Then, there exists $j \in L(\lambda)$ such that $j < j_p(\lambda)$ and $z_j(\lambda) = i_p(\lambda) - \lceil j/2\rceil$. Let $i \in [2k-4]$ such that $f(j) = 2i$. Because $j^p(\lambda) > j$, we have $i \neq i_p(\lambda)$. And since $j \in L(\lambda)$, we obtain $2i = f(j) = 2(\lceil j/2\rceil + z_j(\lambda)) = 2i_p(\lambda)$, which is absurd. So $j_p(\lambda) \notin L(\lambda)$. □

Proof of Proposition 4.3. Let $f \in SP_{k-1}$ and $\lambda = \varphi(f)$ and $g = \phi(\lambda)$. Let $j \in [2k-4]$ and $i \in [k-1]$ such that $f(j) = 2i$.

(1) If $s^j(f)$ is defined in the context (1) of Definition 4.1, then there exists $p \in [m(\lambda)]$ such that $i = i_p(\lambda)$ and $j = j^p(\lambda) = j_p(\lambda)$. Consequently, in view of Lemma 4.6 with $\lambda = \varphi(f)$, we know that $j \notin L(\lambda)$, implying $g(j) = g(j_p(\lambda)) = 2i_p(\lambda) = 2i = f(j)$.
(2) If $s^j(f)$ is defined in the context (2) of Definition 4.1, then $z_j(f) = f(j)/2 - \lceil j/2\rceil = i - \lceil j/2\rceil$. Now it is necessary that $j \in L(\lambda)$: otherwise $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$, and from Lemma 4.6 we would have $j = j_p(\lambda) = j^p(\lambda)$, which is impossible because we are in the context (2) of Definition 4.1. So $j \in L(\lambda)$, implying $g(j) = 2(\lceil j/2\rceil + z_j(\lambda)) = 2(\lceil j/2\rceil + z_j(f)) = 2i = f(j)$.

As a conclusion, we obtain $g = f$ so $\phi \circ \varphi$ is the identity map of SP_{k-1}.
Reciprocally, let $\mu \in IS_k$ and $h = \phi(\mu)$. We are going to prove by induction that $s^j(\mu) = s^j(h)$ for all $j \in [2k-3]$. By definition $s^{2k-3}(\mu) = s^{2k-3}(h) = \varnothing$. Suppose that $s^{j+1}(\mu) = s^{j+1}(h)$ for some $j \in [2k-4]$.

(1) If $s^j(h)$ is defined in the context (1) of Definition 4.1, then there exists $p \in [m(\lambda)]$ such that $h(j) = 2i_p(\mu)$, such that $j = j^p(\mu)$ and such that $s^{j+1}(h)$ is not saturated in $i_p(\mu)$. Since the partial k-shape $s^{j+1}(\mu) = s^{j+1}(h)$ is not saturated in $i_p(\mu)$, by definition $j \geq j_p(\mu)$. Suppose that $j > j_p(\mu)$. Since $j = j^p(\mu)$, we know from Lemma 4.6 (with $\lambda = \mu$ and $f = \phi(\lambda) = h$) that $j_p(\mu) \in L(\mu)$. It means there exists $j' < j_p(\mu) < j$ such that $j' \in L(\mu)$ and $\lceil j'/2\rceil + z_{j'}(\mu) = i_p(\mu)$, implying $h(j') = 2i_p(\mu) = h(j)$, wich contradicts $j = j^p(\mu)$. So $j = j_p(\mu)$, therefore $s^j(\mu)$ is saturated in $i_p(\mu)$. But since we are in the context (1) of Definition 4.1, the partial k-shape $s^j(h)$ is defined as $s^{j+1}(h) \oplus^k_{t(j)} (\lceil (j+1)/2\rceil^{z_j(h)}$ where $z_j(h)$ is the unique integer $z \in [k-1-\lceil j/2\rceil]$ such that $s^{j+1}(h) \oplus^k_{t(j)} (\lceil (j+1)/2\rceil^{z}$ is saturated in $i_p(\mu)$. Since the partial k-shape $s^j(\mu) = s^{j+1}(\mu) \oplus^k_{t(j)} (\lceil (j+1)/2\rceil^{z_j(\mu)} = s^{j+1}(h) \oplus^k_{t(j)} (\lceil (j+1)/2\rceil^{z_j(\mu)}$ is saturated in $i_p(\mu)$, we obtain $z_j(\mu) = z_j(h)$ and $s^j(\mu) = s^j(h)$.
(2) If $s^j(h)$ is defined in the context (2) of Definition 4.1, then $s^j(h) = s^{j+1}(\mu) \oplus^k_{t(j)} \lceil (j+1)/2\rceil^{z_j(h)}$ with $z_j(h) = h(j)/2 - \lceil j/2\rceil$. Now either $h(j) = 2(\lceil j/2\rceil + z_j(\mu))$, in which case we obtain $z_j(h) = z_j(\mu)$, or $h(j) = 2i_p(\mu)$ for some $p \in [m(\mu)]$ such that $j = j_p(\mu) \notin L(\mu))$. In view of Lemma 4.6, it means $j = j^p(\mu)$, which cannot happen because otherwise we would be in the context (1) of Definition 4.1. So $z_j(h) = z_j(\mu)$ and $s^j(h) = s^j(\mu)$.

By induction, we obtain $s^1(\mu) = s^1(h)$, thence $\mu = \varphi(h)$. Consequently, the map $\varphi \circ \phi$ is the identity map of IS_k. □

5. Extensions

Dumont and Foata [DF76] introduced a refinement of Gandhi polynomials $(Q_{2k}(x))_{k\geq 1}$ through the polynomial sequence $(F_k(x,y,z))_{k\geq 1}$ defined by $F_1(x,y,z)=1$ and

$$F_{k+1}(x,y,z)=(x+y)(x+z)F_k(x+1,y,z)-x^2F_k(x,y,z).$$

Note that $x^2F_k(x,1,1)=Q_{2k}(x)$ for all $k\geq 1$ in view of Formula 1. Now, for all $k\geq 2$ and $f\in SP_k$, let $max(f)$ be the number of *maximal* points of f (points $j\in[2k-2]$ such that $f(j)=2k$) and $pro(f)$ the number of *prominent* points (points $j\in[2k-2]$ such that $f(i)<f(j)$ for all $i\in[j-1]$). For example, if f is the surjective pistol $(2,4,4,8,8,6,8,8)\in SP_4$ depicted in Figure 1, then the maximal points of f are $\{4,5\}$, and its prominent points are $\{2,4\}$. Dumont and Foata gave a combinatorial interpretation of $F_k(x,y,z)$ in terms of surjective pistols.

Theorem 5.1 ([DF76]). *For all $k\geq 2$, the Dumont-Foata polynomial $F_k(x,y,z)$ is symmetrical, and is generated by SP_k:*

$$F_k(x,y,z)=\sum_{f\in SP_k} x^{max(f)}y^{fix(f)}z^{pro(f)}.$$

In 1996, Han [Han96] gave another interpretation by introducing the statistic $sur(f)$ defined as the number of *surfixed* points of $f\in SP_k$ (points $j\in[2k-2]$ such that $f(j)=j+1$; for example, the surfixed points of the surjective pistol $f\in SP_4$ of Figure 1 are $\{1,3\}$).

Theorem 5.2 ([Han96]). *For all $k\geq 2$, the Dumont-Foata polynomial $F_k(x,y,z)$ has the following combinatorial interpretation:*

$$F_k(x,y,z)=\sum_{f\in SP_k} x^{max(f)}y^{fix(f)}z^{sur(f)}.$$

Theorem 2.1 then appears as a particular case of Theorem 5.1 or Theorem 5.2 by setting $x=z=1$ (and by applying the symmetry of $F_k(x,y,z)$). Furthermore, for all $f\in SP_k$ and $j\in[2k-2]$, we say that j is a *lined* point of f if there exists $j'\in[2k-2]\backslash\{j\}$ such that $f(j)=f(j')$. We define $mo(f)$ (resp. $me(f)$) as the number of odd (resp. even) maximal points of f, and $fl(f)$ (resp. $fnl(f)$) as the number of lined (resp. non lined) fixed points of f, and $sl(f)$ (resp. $snl(f)$) as the number of lined (resp. non lined) surfixed points of f. Dumont [Dum95] defined generalized Dumont-Foata polynomials $(\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z}))_{k\geq 1}$ by

$$\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z})=\sum_{f\in SP_k} x^{mo(f)}y^{fl(f)}z^{snl(f)}\bar{x}^{me(f)}\bar{y}^{fnl(f)}\bar{z}^{sl(f)}.$$

This a refinement of Dumont-Foata polynomials, considering $\Gamma_k(x,y,z,x,y,z)=F_k(x,y,z)$. Dumont conjectured the following induction formula: $\Gamma_1(x,y,z,\bar{x},\bar{y},\bar{z})=1$ and

$$\begin{aligned}\Gamma_{k+1}(x,y,z,\bar{x},\bar{y},\bar{z})=(x+\bar{z})(y+\bar{x})\Gamma_k(x+1,y,z,\bar{x}+1,\bar{y},\bar{z})\\ +(x(\bar{y}-y)+\bar{x}(z-\bar{z})-x\bar{x})\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z}).\end{aligned}\tag{3}$$

This was proven independently by Randrianarivony [Ran94] and Zeng [Zen96]. See also [JV11] for a new combinatorial interpretation of $\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z})$.

Now, let $f\in SP_{k-1}$ and $\lambda=\varphi(f)\in IS_k$. For all $j\in[2k-4]$, we say that j is a *chained* k-site of λ if $j\notin L(\lambda)$. Else, we say that it is an *unchained* k-site. In view of Lemma 4.6, an integer $j\in[2k-4]$ is a chained k-site if and only if $j=j_p(\lambda)=j^p(\lambda)$ for some $p\in[m(\lambda)]$, in which case $f(j)=2i_p(\lambda)$ (the integer j is forced to be mapped to $2i_p(\lambda)$, thence the use of the word *chained*). If j is an unchained k-site, by definition $f(j)=2(\lceil j/2\rceil+z_j(\lambda))$. Consequently, every statistic of Theorems 5.1, 5.2 and Formula 3 has its own equivalent among irreducible k-shapes. However, the objects counted by these statistics are not always easily pictured or

formalized. We only give the irreducible k-shapes version of Theorem 5.2.
Recall that for all $i \in [k-2]$, the integer $2i$ is a fixed point of f if and only if $2i$ is a free k-site of λ, which is also equivalent to $z_{2i}(\lambda) = 0$. We extend the notion of free k-site to any $j \in [2k-4]$: the integer j is said to be a free k-site if $z_j(\lambda) = 0$. Notice that free k-sites of λ are necessarily unchained because $z_j(\lambda) = 0$ implies $s^j(\lambda) = s^{j+1}(\lambda)$ thence $j \neq j_p(\lambda)$ for all $p \in [m(\lambda)]$. We denote by $fro(\lambda)$ the quantity of odd free sites of λ. We denote by $ful(\lambda)$ the quantity of *full* k-site of λ (namely, unchained k-sites $j \in L(\lambda)$ such that $z_j(\lambda) = k-1-\lceil j/2 \rceil$), and by $sch(\lambda)$ the quantity of *surchained* k-sites (chained k-sites $j \in [2k-4]$ such that $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$ such that $2i_p(\lambda) = j+1$). Theorem 5.2 can now be reformulated as follows.

Theorem 5.3. *For all $k \geq 2$, the Dumont-Foata polynomial $F_k(x,y,z)$ has the following combinatorial interpretation:*

$$F_k(x,y,z) = \sum_{\lambda \in IS_{k+1}} x^{ful(\lambda)} y^{fr(\lambda)} z^{fro(\lambda)+sch(\lambda)}.$$

Proof. First of all, maximal points of f are full k-sites of λ: if $f(j) = 2k-2$ then $z_j(f)$ is necessarily defined in the context (2) of Definition 4.1, thence $z_j(\lambda) = z_j(f) = f(j)/2 - \lceil j/2 \rceil = k-1-\lceil j/2 \rceil$, and $j \in L(\lambda)$ because otherwise $f(j)$ would equal $2i_p(\lambda) < 2k-2$ for some $p \in [m(\lambda)]$. So j is a full k-site of λ. Reciprocally, if $j \in L(\lambda)$ is such that $z_j(\lambda) = k-1-\lceil j/2 \rceil$, then $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda)) = 2k-2$ so j is a maximal point of f. Afterwards, surfixed points of f are odd free k-sites and surchained k-sites of λ: if $f(j) = j+1$, then $j = 2i-1$ for some $i \in [k-1]$ and either $j \in L(\lambda)$, in which case $f(j) = 2(i + z_j(\lambda)) = 2i$ thence j is an odd free k-site, or $j = j_p(\lambda) = j^p(\lambda)$ for some $p \in [m(\lambda)]$ such that $2i_p(\lambda) = 2i = j+1$, *i.e.*, the integer j is a surchained k-site. Reciprocally, if j is an odd free k-site then $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda)) = 2(\lceil j/2 \rceil) = j+1$, and if j is a surchained k-site then in particular $f(j) = 2i_p(\lambda) = j+1$ for some $p \in [m(\lambda)]$. As a conclusion, the result comes from Theorem 2.2. $\square$

Aknowledgement

I thank Jiang Zeng for his comments and useful references.

References

[Car71] Leonard Carlitz. A conjecture concerning Genocchi numbers. *Norske Vid. Selsk. Skr. (Trondheim)*, 9:4, 1971.

[DF76] Dominique Dumont and Dominique Foata. Une propriété de symétrie des nombres de genocchi. *Bull. Soc. Math. France*, 104:433–451, 1976.

[DR94] Dominique Dumont and Arthur Randrianarivony. Dérangements et nombres de Genocchi. *Disc. Math.*, 132:37–49, 1994.

[Dum72] Dominique Dumont. Sur une conjecture de Gandhi concernant les nombres de Genocchi. *Disc. Math.*, 1:321–327, 1972.

[Dum74] Dominique Dumont. Interprétations combinatoires des nombres de Genocchi. *Duke Math. J.*, 41:305–318, 1974.

[Dum95] Dominique Dumont. Conjectures sur des symétries ternaires liées aux nombres de Genocchi. In FPSAC 1992, editor, *Discrete Math.*, number 139, pages 469–472, 1995.

[Han96] Guo Niu Han. Symétries trivariées sur les nombres de genocchi. *Europ. J. Combinatorics*, 17:397–407, 1996.

[HM11] Florent Hivert and Olivier Mallet. Combinatorics of k-shape and Genocchi numbers. In FPSAC 2011, editor, *Discrete Math. Theor. Comput. Sci. Proc.*, pages 493–504, Nancy, 2011.

[JV11] Matthieu Josuat-Vergès. Dumont-Foata polynomials and alternative tableaux. Sém. Lothar. Combin. 64, Art. B64b, 17pp, 2011.

[LLMS13] Thomas Lam, Luc Lapointe, Jennifer Morse, and Marc Shimozono. The poset of k-shapes and branching rules for k-schur functions. *Mem. Amer. Math. Soc.*, 223(1050), 2013.

[Mal11] Olivier Mallet. Combinatoire des k-formes et nombres de Genocchi. Séminaire de combinatoire et théorie des nombres de l'ICJ, may 2011.

[OEI] OEIS Foundation Inc. (2011). The On-Line Encyclopedia of Integer Sequences. `http://oeis.org/A110501`.

[Ran94] Arthur Randrianarivony. Polynômes de Dumont-Foata généralisés. Sém. Lothar. Combin. 32, Art. B32d, 12pp, 1994.

[RS73] John Riordan and Paul R. Stein. Proof of a conjecture on Genocchi numbers. *Discrete Math.*, 5:381–388, 1973.

[Sta99] R.P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press,Cambridge, 1999.

[Zen96] Jiang Zeng. Sur quelques propriétés de symétrie des nombres de Genocchi. *Disc. Math.*, 153:319–333, 1996.

Institut Camille Jordan, Université Claude Benard Lyon 1 (France)
E-mail address: `bigeni@math.univ-lyon1.fr`